Gradient-Like Flows and Self-Indexing in Stratified Morse Theory

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Abstract

We develop the idea of self-indexing and the technology of gradient-like vector fields in the setting of Morse theory on a complex algebraic stratification. Our main result is the local existence, near a Morse critical point, of gradient-like vector fields satisfying certain "stratified dimension bounds up to fuzz" for the ascending and descending sets. As a global consequence of this, we derive the existence of self-indexing Morse functions.

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1 Introduction

The goal of this paper is to develop the idea of self-indexing and the technology of gradient-like vector fields in the setting of Morse theory on a complex algebraic stratification. Our main result (Theorem 1.5) is the local existence, near a Morse critical point, of gradient-like vector fields satisfying certain "stratified dimension bounds up to fuzz" for the ascending and descending sets. As a global consequence of this, we derive the existence of self-indexing Morse functions (Theorem 1.6).

This paper traces its roots to the informal lecture notes "Intersection Homology and Perverse Sheaves" by R. MacPherson [9]. These notes outline a vision for developing the theory of middle perversity perverse sheaves on a stratified complex variety, starting with a definition of a perverse sheaf in the style of the Eilenberg-Steenrod axioms. This approach to perverse sheaves relies on self-indexing Morse functions as a key technical tool. We now proceed to introduce this circle of ideas at a leisurely pace.

1.1 Classical Morse Theory

Let X^d be a compact smooth manifold, and let $f : X \to \mathbb{R}$ be a Morse function. Write Σ_f for the set of critical points of X. The function f is called self-indexing if $f(p) = \mathrm{index}_f(p)$ for every $p \in \Sigma_f$. The significance of this definition is the following. A self-indexing f gives rise canonically to a cochain complex $\mathcal{C}_f = (C^*, d)$ of $\mathbb{C}$-vector spaces such that:

(a) C^i has a natural basis up to sign parameterized by $\{\, p \in \Sigma_f \mid \mathrm{index}_f(p) = i \,\}$;

(b) $H^*(\mathcal{C}_f) = H^*(X; \mathbb{C})$.

Often, one uses an arbitrary Morse function plus another piece of structure (e.g., a generic metric) to define a complex for computing $H^*(X; \mathbb{C})$. However, one needs self-indexing to produce a canonical complex starting with a Morse function alone. The complex $\mathcal{C}_f$ is called the Morse-Smale complex. It plays a central role in Milnor's beautiful exposition [11] of Smale's proof of the h-cobordism theorem in dimensions > 4. The first step of that proof is the following theorem.

Theorem 1.1 [16, Theorem B] *Every compact, smooth manifold X admits a self-indexing Morse function $f : X \to \mathbb{R}$.*

We recall an outline of the proof of this in [11]. Start with any Morse function $g : X \to \mathbb{R}$. A ∇g-like vector field V on X is a C^∞ vector field such that:

(a) $V_p = 0$ for all $p \in \Sigma_g$;

(b) $V_x\, g > 0$ for all $x \notin \Sigma_g$.

(This definition is slightly different from the one in [11].) Given such a V, let $\psi_V : X \times \mathbb{R} \to X$ denote its flow. Then for $p \in \Sigma_g$, let

$$M_V^{\pm}(p) = \{\, x \in X \mid \lim_{t \to \mp\infty} \psi_V(x, t) = p \,\}.$$

The sets $M_V^{\pm}(p)$ are called the ascending and descending sets of p (relative to V). Here now are the three main steps of the proof of Theorem 1.1.

Step 1: Morse lemma. By Morse lemma, there exists a ∇g-like vector field V such that, for every $p \in \Sigma_g$, the set $M_V^-(p)$ is a manifold of dimension $\mathrm{index}_g(p)$, and $M_V^+(p)$ is a manifold of dimension $d - \mathrm{index}_g(p)$.

Step 2: General position. By perturbing the vector field V, we can ensure that $M_V^-(p) \cap M_V^+(q) = \emptyset$ whenever $\mathrm{index}_g(p) \leq \mathrm{index}_g(q)$.

Step 3: Modifying g. There exists a Morse function $f : X \to \mathbb{R}$ such that:

(a) near each $p \in \Sigma_g$, we have $f(x) = g(x) + \mathrm{index}_g(p) - g(p)$;

(b) V is ∇f-like.

Conditions (a) and (b) guarantee that f is self-indexing. The role of the vector field V is to ensure that $\Sigma_f = \Sigma_g$.

The main result of this paper (Theorem 1.5) is a replacement for Step 1 in the complex stratified setting. Steps 2 and 3 carry over in a more or less straightforward fashion.

1.2 Stratified Morse Theory

Let now X^d be a smooth complex algebraic variety, and let $\mathcal{S}$ be an algebraic Whitney stratification of X. Stratified Morse theory, pioneered by Goresky and MacPherson (see [3], [4]), aims to study the topology of the pair $(X, \mathcal{S})$ by means of a real C^∞ function $f : X \to \mathbb{R}$. The following definitions go back to the original paper [3].

Definition 1.2 *(i) Let $p \in X$ be a point contained in a stratum S. We say that p is critical for f ($p \in \Sigma_f$) if it is critical for the restriction $f|_S$.*

(ii) For $S \in \mathcal{S}$, let Λ_S be the conormal bundle $T^*_S X \subset T^*X$, and let $\Lambda = \Lambda_{\mathcal{S}} = \bigcup_S \Lambda_S$. This Λ is called the conormal variety to $\mathcal{S}$; it is a closed Lagrangian subvariety of T^*X (the fact that Λ is closed is equivalent to one of the Whitney conditions). Note that $p \in \Sigma_f$ if and only if $d_p f \in \Lambda$.

(iii) Let $p \in \Sigma_f$. We say that p is Morse for f if it is Morse for the restriction $f|_S$ (where S is the stratum containing p) and $d_p f \in \Lambda^0$, the smooth part of Λ. The set Λ^0 is called the set of generic conormal vectors to $\mathcal{S}$.

It is important to note that there is nothing specifically complex about the above definitions: they would apply equally well to any Whitney stratification of a real C^∞ manifold. By contrast, the following definition is only justified in the complex algebraic (or analytic) setting.

Definition 1.3 *Let $p \in \Sigma_f$ be Morse and let S be the stratum of p. We define*

$$\mathrm{index}_f(p) = \mathrm{index}_{f|_S}(p) - \dim_{\mathbb{C}} S = \mathrm{index}_{f|_S}(p) + \mathrm{codim}_{\mathbb{C}} S - d.$$

The normalization constant $d = \dim_{\mathbb{C}} X$ is subtracted to make the possible range of values of the index symmetric about the origin: $\mathrm{index}_f(p) \in \{-d, \dots, d\}$. Ignoring this normalization, the index is attempting to count the (real) "descending directions" of f at p, by counting the tangent directions in the obvious way and then postulating that exactly half of the normal directions are descending. The point of this paper is that this method of counting is justified. The justification comes from a local result concerning Morse theory near a point stratum. Before stating the result itself, we present a somewhat sharper conjecture.

Conjecture 1.4 *Let $X = \mathbb{C}^d$, $p \in X$ be the origin, and $\mathcal{S}$ be an algebraic Whitney stratification of X such that $\{p\}$ is a stratum. Let $\Lambda^0_p = \Lambda_{\{p\}} \cap \Lambda^0$ be the set of generic covectors at p, and let $f \in \Lambda^0_p$. Regard f as a linear function $f : X \to \mathbb{R}$. Then there exists a closed ball B around p and an $\mathcal{S}$-preserving ∇f-like vector field V on B such that the ascending and descending sets $M^{\pm}_V(p)$ satisfy:*

$$\dim_{\mathbb{R}} M^{\pm}_V(p) \cap S \le \dim_{\mathbb{C}} S \quad \textit{for every} \quad S \in \mathcal{S}.$$

The reader is referred to Section 2.2 for precise definitions of an $\mathcal{S}$-preserving ∇f-like vector field and of the sets $M^{\pm}_V(p)$. However, the truth and the level of difficulty of this conjecture might be sensitive to the exact class of vector fields chosen. Therefore, it is best to view the conjecture as being somewhat imprecise, with the phrase "$\mathcal{S}$-preserving ∇f-like vector field" open to interpretation. Theorem 1.5 should be seen as Conjecture 1.4 "up to fuzz".

Theorem 1.5 *Let X, p, $\mathcal{S}$ be as in Conjecture 1.4, and let $\Delta \subset \Lambda^0_p$ be an open set. Then there exit an $f \in \Delta$ (which we view as a linear function $f : X \to \mathbb{R}$), a closed ball B around p, and a closed, real semi-algebraic $K \subset B$ such that:*

(i) $\dim_{\mathbb{R}} K \cap S \le \dim_{\mathbb{C}} S$ for every $S \in \mathcal{S}$;

(ii) $f^{-1}(0) \cap K = \{p\}$; and

(iii) for every open $\mathcal{U} \supset K$, there exists an $\mathcal{S}$-preserving ∇f-like vector field V on B with $M^{\pm}_V(p) \subset \mathcal{U}$.

Theorem 1.5 will be proved in Sections 4 and 5. It is important to note that for both Conjecture 1.4 and Theorem 1.5 it is crucial that the pair $(X, \mathcal{S})$ is assumed to be complex. For contrast, consider $X = \mathbb{R}^3$, let $\mathcal{S}$ be the obvious stratification with six strata whose 2-skeleton is the cone $x^2 + y^2 = z^2$, and take $f(x, y, z) = z$. Then every $\mathcal{S}$-preserving ∇f-like vector field V on X will have $\dim M^+_V(p) = \dim M^-_V(p) = \dim X = 3$. (An ice cream cone can hold a 3d amount of ice cream!) This example shows that the concept of index does not generalize well to Morse theory on a real stratification. It is also easy to modify the above example to one with all strata of even dimension (just put the same cone in $\mathbb{R}^4$). The following consequence of Theorem 1.5 will be proved in Section 6.

Theorem 1.6 *Every proper, Whitney stratified complex algebraic variety $(X, \mathcal{S})$ admits a self-indexing Morse function $f : X \to \mathbb{R}$.*

1.3 Perverse Sheaves

We now explain the motivation behind Theorems 1.5 and 1.6 coming from the theory of middle perversity perverse sheaves. Let $(X^d, \mathcal{S})$ be as in Section 1.2. Associated to the pair $(X, \mathcal{S})$ is the category $\mathcal{P}(X, \mathcal{S})$ of middle perversity perverse sheaves on X constructible with respect to $\mathcal{S}$. This category was first introduced by Beilinson, Bernstein, and Deligne in the seminal paper [1] in 1982. They defined $\mathcal{P}(X, \mathcal{S})$ as a full subcategory of $D^b_{\mathcal{S}}(X)$, the bounded $\mathcal{S}$-constructible derived category of sheaves on X. This way $\mathcal{P}(X, \mathcal{S})$ inherits an additive structure from $D^b_{\mathcal{S}}(X)$. In fact, it turns out to be an abelian category, unlike the derived category $D^b_{\mathcal{S}}(X)$. Perverse sheaves are "simpler" than the derived category in several other respects. For example, they form a stack (i.e., objects and morphisms are locally defined), which is not the case for $D^b_{\mathcal{S}}(X)$.

For these reasons, it was long felt desirable to have a definition of $\mathcal{P}(X, \mathcal{S})$ which does not rely on the more complicated derived category, and which elucidates the very simple formal properties of perverse sheaves. Such a definition was proposed by MacPherson in his AMS lectures in San Francisco in January 1991. Informal notes of those lectures [9] produced by MacPherson (and dated 1990) were distributed at the meeting, but were never published. A slightly modified version of the same definition has now appeared in print in the lecture notes [5] from the summer institute held in Park City, Utah in July 1997.

Roughly, MacPherson's definition (the 1997 version) proceeds as follows. A perverse sheaf $P \in \mathcal{P}(X, \mathcal{S})$ is an assignment $(Y, Z) \mapsto H^*(Y, Z; P)$, plus coboundary and pull-back maps, subject to a set of axioms modeled after the Eilenberg-Steenrod axioms. Here (Y, Z) is a *standard pair* in X, that is a pair of compact subsets of a certain special kind. Namely, Y is a smooth, real $2d$ dimensional submanifold of X with corners of codimension two, and Z is a union of some of the walls of Y, so that each corner is formed by one wall from Z and one wall not from Z, and all the boundary strata of Y are transverse to $\mathcal{S}$. It is a very pleasant feature of this definition that we only need to consider standard pairs. The structure group $H^*(Y, Z; P)$ is called the cohomology of Y relative to Z with coefficients in P; it is a finite dimensional, graded vector space over $\mathbb{C}$.

Turning now to the axioms, all but one (the dimension axiom) represent a more or less straightforward adaptation of the classical Eilenberg-Steenrod to the setting of pairs inside a fixed space. The remaining axiom runs as follows.

Dimension Axiom: Let $f : X \to \mathbb{R}$ be a proper Morse function, and $c \in \mathbb{R}$ be a value such that $f^{-1}(c)$ contains exactly one critical point: $f^{-1}(c) \cap \Sigma_f = \{p\}$. Then

$$H^i(\{x \in X \mid c - \epsilon \leq f(x) \leq c + \epsilon\}, \{x \in X \mid f(x) = c - \epsilon\}; P) = 0$$

for $i \neq \operatorname{index}_f(p)$ and $0 < \epsilon \ll 1$.

Here is how this definition fits with the idea of self-indexing. Given a standard pair (Y, Z), one can use a variant of Theorem 1.6 to find a self-indexing function

$$f : Y \setminus \{\text{corners}\} \to [-d-1, d+1]$$

which is *adapted to* the pair (Y, Z). More precisely, this means that $f^{-1}(-d-1) = Z \setminus \{\text{corners}\}$, $f^{-1}(d+1) = \partial Y \setminus Z$, and the level sets of f "look like pages of a book" near the corners. Now, just as in classical Morse theory, such a function f gives rise to a functor

$$\mathcal{C}_f : \mathcal{P}(X, \mathcal{S}) \to \{\text{cochain complexes}\}$$

with the property that $H^*(\mathcal{C}_f(P)) = H^*(Y, Z; P)$ for every $P \in \mathcal{P}(X, \mathcal{S})$. If we write $\mathcal{C}_f = (C^*, d)$, then the functor

$$C^i : \mathcal{P}(X, \mathcal{S}) \to \{\text{vector spaces}\}$$

breaks up into a direct sum over $\{\, p \in \Sigma_f \mid \text{index}_f(p) = i \,\}$ of the so-called *Morse group* functors

$$M_{d_p f} : \mathcal{P}(X, \mathcal{S}) \to \{\text{vector spaces}\}.$$

Each of the $M_{d_p f}$ is localized near p, i.e., factors through the restriction to any open neighborhood of p. (Perverse sheaves on open subsets of X and restriction functors between them are defined in the obvious way.) Moreover, as implied by the notation, the functor $M_{d_p f}$ depends on f only through the differential $d_p f \in \Lambda^0$.

The existence of the lift $\mathcal{C}_f$ from cohomology to cochain complexes with the above properties, by itself, has important consequences for the structure of the category $\mathcal{P}(X, \mathcal{S})$. For example, it immediately implies that morphisms of perverse sheaves are locally defined. More is true however: the functor $\mathcal{C}_f$ is, in some sense, independent of f; the only essential dependence is on the pair (Y, Z). To be precise, given any two self-indexing functions f_1, f_2 adapted to (Y, Z), the functors $\mathcal{C}_{f_1}$ and $\mathcal{C}_{f_2}$ are related by a quasi-isomorphism which is itself canonical up to chain homotopy. A reader familiar with Floer homology has, no doubt, anticipated this. However, unlike in Floer homology, this independence of the function requires a proof which is rather more involved than the proof of $d^2 = 0$. (The later is a straightforward consequence of the axioms.)

Once the independence of the function is established in a suitably flexible form, all of the (soft) formal properties of perverse sheaves can be derived from considering the complexes $\mathcal{C}_f$ for different functions and standard pairs. For example, the abelian property of $\mathcal{P}(X, \mathcal{S})$ follows easily from the abelian property of the category of cochain complexes. Finally, one can develop the theory of perverse sheaves, starting with MacPherson's definition, to the point of proving the following.

Theorem 1.7 *MacPherson's definition of $\mathcal{P}(X, \mathcal{S})$ agrees with the original definition due to Beilinson-Bernstein-Deligne.*

The present author has recently had an opportunity to work through a proof of Theorem 1.7 in a graduate course at MIT, and a future paper describing this proof is being planned. The goal of the present paper is to provide the main geometric tool: self-indexing Morse functions. We conclude this introduction with a conjecture due to MacPherson.

Conjecture 1.8 *Omitting the dimension axiom from MacPherson's definition of $\mathcal{P}(X, \mathcal{S})$ gives a category which is equivalent to $D^b_{\mathcal{S}}(X)$.*

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2 Technical Preliminaries

In this section, we summarize the preliminary material on controlled and weakly controlled vector fields that we will need, and define precisely the ingredients of Theorem 1.5. The reader is referred to [2, Chapter 2] and [14, Chapter 2.5] for a detailed treatment of controlled vector fields. However, our definitions differ slightly from these sources. The main distinction is that our notion of a quasi-distance function is more flexible than the corresponding notions in [2] and [14]. This is necessary to make Theorem 3.1 true "in the controlled category."

2.1 Controlled Vector Fields

Let X be a real C^∞ manifold with a Whitney stratification $\mathcal{S}$.

Definition 2.1 *Let $S \in \mathcal{S}$ be a stratum, and let U_S be an open neighborhood of S. A tubular projection $\Pi_S : U_S \to S$ is a smooth submersion restricting to the identity on S.*

Definition 2.2 *Let M be a smooth manifold, and let $E \to M$ be a vector bundle with zero section Z. A quasi-norm on E is a smooth function $\rho : E \setminus Z \to \mathbb{R}_+$ such that $\rho(\lambda\, e) = \lambda\, \rho(e)$ for every $e \in E \setminus Z$ and $\lambda \in \mathbb{R}_+$.*

Definition 2.3 *Let $S \in \mathcal{S}$ be a stratum, and let U_S be an open neighborhood of S. A quasi-distance function $\rho : U_S \setminus S \to \mathbb{R}_+$ is a smooth function satisfying the following condition. There exist a vector bundle $p : E \to S$ with zero section Z, an open neighborhood $U' \subset E$ of Z, and a diffeomorphism $\phi : U' \to U$, such that $\phi|_Z = p|_Z$ and $\rho \circ \phi$ is the restriction to $U' \setminus Z$ of a quasi-norm on E.*

Definition 2.4 *Control data on $(X, \mathcal{S})$ is a collection $\{U_S, \Pi_S, \rho_S\}_{S \in \mathcal{S}}$, where $U_S \supset S$ is an open neighborhood, $\Pi_S : U_S \to S$ is a tubular projection, and $\rho_S : U_S \setminus S \to \mathbb{R}_+$ is a quasi-distance function, subject to the compatibility conditions $\Pi_S \circ \Pi_T = \Pi_S$ and $\rho_S \circ \Pi_T = \rho_S$, both being equalities between maps of $U_S \cap U_T$.*

Definition 2.5 *Let $\{U_S, \Pi_S, \rho_S\}$ be control data on $(X, \mathcal{S})$, let $\mathcal{U} \subset X$ be an open subset, let A be a set, and let $f : \mathcal{U} \to A$ be a map of sets. We say that $\{U_S, \Pi_S, \rho_S\}$ is f-compatible on $\mathcal{U}$ if, for every $S \in \mathcal{S}$, there is a neighborhood U'_S of $S \cap \mathcal{U}$ such that $f \circ \Pi_S = f$ on U'.*

Definition 2.6 *Let $(X, \mathcal{S})$, $(\hat{X}, \hat{\mathcal{S}})$ be two Whitney stratified manifolds, with control data $\{U_S, \Pi_S, \rho_S\}$ and $\{U_{\hat{S}}, \Pi_{\hat{S}}, \rho_{\hat{S}}\}$. A controlled homeomorphism $\phi : X \to \hat{X}$ is a homeomorphism which takes strata diffeomorphicly onto strata, establishing a bijection $S \mapsto \hat{S}$, and satisfies the following condition. For every $S \in S$, there is a neighborhood $U'_S \subset U_S$ of S such that $\phi \circ \Pi_S = \Pi_{\hat{S}} \circ \phi$ and $\phi \circ \rho_S = \rho_{\hat{S}} \circ \phi$ on U'_S.*

Definition 2.7 *Suppose we are given control data $\{U_S, \Pi_S, \rho_S\}$ on $(X, \mathcal{S})$. A controlled vector field V on X compatible with $\{U_S, \Pi_S, \rho_S\}$ is a collection $\{V_S\}_{S \in \mathcal{S}}$ of smooth vector fields on the individual strata, satisfying the following condition. For every $S \in \mathcal{S}$, there exists a neighborhood $U'_S \subset U_S$ of S such that:*
(a) $(\Pi_S)_* V_x = V_{\Pi_S(x)}$ for every $x \in U'_S$;
(b) $V_x \, \rho_S = 0$ for every $x \in U'_S \setminus S$.

Integrating controlled vector fields is a basic technique for constructing controlled homeomorphisms, going back to the work of Thom [17] and Mather [10]. The following variant of [15, Lemma 4.11] and [13, Theorem 1.1] will serve as our basic tool for constructing controlled vector fields.

Lemma 2.8 *Let $\{U_S, \Pi_S, \rho_S\}$ be control data on $(X, \mathcal{S})$. Let S be a stratum, let $U \subset S$ be open in S, and let V be a smooth vector field on U. Then there exist an open $\mathcal{U} \subset X$ with $\mathcal{U} \cap S = U$ and a controlled vector field $\tilde{V}$ on $\mathcal{U}$, compatible with $\{U_S, \Pi_S, \rho_S\}$, such that $\tilde{V}|_U = V$. Furthermore, the vector field $\tilde{V}$ can be chosen to be continuous as a section of $T\mathcal{U}$.* □

2.2 Weakly Controlled Vector Fields

Controlled vector fields are not suitable for discussing the ascending and descending sets. Indeed, the trajectory of a controlled vector field can not approach a point on a smaller stratum as time tends to infinity. We will therefore need the following definition.

Definition 2.9 *Suppose we are given control data* $\{U_S, \Pi_S, \rho_S\}$ *on* $(X, \mathcal{S})$. *A weakly controlled vector field* V *on* X *is a collection* $\{V_S\}_{S\in\mathcal{S}}$ *of smooth vector fields on the individual strata, satisfying the following condition. For every* $S \in \mathcal{S}$, *there exists a neighborhood* $U'_S \subset U_S$ *of* S *and a number* $k > 0$ *such that:*

(a) $(\Pi_S)_* V_x = V_{\Pi_S(x)}$ *for every* $x \in U'_S$;

(b) $|V_x\, \rho_S| < k \cdot \rho_S(x)$ *for every* $x \in U'_S \setminus S$.

Proposition 2.10 *Weakly controlled vector fields integrate to stratum preserving homeomorphisms. More precisely, let* V *be a weakly controlled vector field on* X. *Then for every* $x \in X$, *there is a neighborhood* U_x *of* x *and number* $t_0 > 0$ *such that, for every* $t \in [-t_0, t_0]$, *the time-*t *flow of* V *is defined on* U_x *and gives a stratum preserving homeomorphism* $\psi_{V,t} : U_x \to \psi_{V,t}(U_x)$, *which is smooth on each stratum.*

Proof: The main thing to check is that a trajectory of V lying in a stratum T can not approach a point on a smaller stratum $S \subset \bar{T}$ in finite time. This follows from condition (b) in Definition 2.9. See [14, Proposition 2.5.1] for more details. $\square$

The next two definitions clarify the statement of Theorem 1.5.

Definition 2.11 *Let* $f : X \to \mathbb{R}$ *be a* C^∞ *function. An* $\mathcal{S}$*-preserving* ∇f*-like vector field* V *on an open subset* $\mathcal{U} \subset X$ *is a weakly controlled vector field (compatible with some control data on* $(X, \mathcal{S})$*) satisfying:*

(a) $V_p = 0$ *for all* $p \in \Sigma_f \cap \mathcal{U}$;

(b) $V_x\, g > 0$ *for all* $x \in \mathcal{U} \setminus \Sigma_f$.

A ∇f*-like vector field on an arbitrary subset* $A \subset X$ *(e.g., on a closed ball) is the restriction to* A *of a* ∇f*-like vector field on some open* $\mathcal{U} \supset A$.

Definition 2.12 *Let* $f : X \to \mathbb{R}$ *be a* C^∞ *function, let* V *be a* ∇f*-like vector field on some* $A \subset X$, *and let* $p \in \Sigma_f \cap A$. *We define* $M_V^-(p)$ *to be the set of all* $x \in A$ *such that the trajectory* $\psi_{V,t}(x)$ *is contained in* A *for all* $t \geq 0$ *and we have* $\lim_{t\to\infty} \psi_{V,t}(x) = p$. *The ascending set* $M_V^+(p)$ *is defined similarly.*

2.3 The Flow Topology

In this section, we discuss the notion of the flow topology on the set of weakly controlled vector fields. It will give us a degree of flexibility, making some of our constructions less tied to the choice of control data.

Let $\mathcal{V}(X, \mathcal{S})$ be the set of all weakly controlled vector fields on $(X, \mathcal{S})$, compatible with all possible control data, and let $\mathcal{V}(X)$ be the union of the $\mathcal{V}(X, \mathcal{S})$ over all Whitney stratifications $\mathcal{S}$ of X. Fix a Riemannian metric g on X. Let $V \in \mathcal{V}(X)$, let $K \subset X$ be a compact subset, let $t > 0$ be a number such that $\psi_{V,s}(x)$ is defined for all $x \in K$ and all $s \in [-t, t]$, and let $\epsilon > 0$ be any positive number. Define

$$\mathcal{U}(V, K, t, \epsilon) = \{V' \in \mathcal{V} \mid \forall x \in K, s \in [-t, t] : \mathrm{dist}_g(\psi_{V,s}(x),\, \psi_{V',s}(x)) < \epsilon\},$$

where we set $\mathrm{dist}_g(\psi_{V,s}(x),\, \psi_{V',s}(x)) = +\infty$ if $\psi_{V',s}(x)$ is undefined.

Definition 2.13 *The flow topology on $\mathcal{V}(X)$ is the weakest topology in which all the $\mathcal{U}(V, K, t, \epsilon)$ are open sets.*

It is easy to see that the flow topology is independent of the metric g.

Proposition 2.14 *(i) Let $\mathcal{S}$ and $\hat{\mathcal{S}}$ be two Whitney stratifications of X such that $\hat{\mathcal{S}}$ is a refinement of $\mathcal{S}$. Fix a vector field $\hat{V} \in \mathcal{V}(X, \hat{\mathcal{S}})$ and control data $\{U_S, \Pi_S, \rho_S\}$ on $(X, \mathcal{S})$. Then there exists a sequence $\{V_i \in \mathcal{V}(X, \mathcal{S})\}_{i\in\mathbb{N}}$ of vector fields compatible with $\{U_S, \Pi_S, \rho_S\}$ such that $V_i \to \hat{V}$ in the flow topology.*

(ii) In the situation of part (i), assume that $\hat{\mathcal{S}}$ has a unique point stratum $\{p\}$, and that $f : X \to \mathbb{R}$ is a smooth function whose only stratified critical point with respect to $\hat{\mathcal{S}}$ is p. Assume also that V is ∇f-like. Then all the V_i can be chosen to be ∇f-like too.

Proof: This is an exercise using Lemma 2.8 and partitions of unity. For part (i), the first step is to show that there is a sequence $\{\hat{V}_i \in \mathcal{V}(X, \hat{\mathcal{S}})\}$ of continuous vector fields compatible with the same control data as V, such that $\hat{V}_i \to \hat{V}$ in the flow topology. The second step is to show that each $\hat{V}_i$ can be approximated in the C^0 topology by continuous vector fields from $\mathcal{V}(X, \mathcal{S})$ compatible with $\{U_S, \Pi_S, \rho_S\}$. Then it remains to note that the C^0 topology on continuous weakly controlled vector fields is stronger than the flow topology. Part (ii) is similar. $\square$

3 Stratified Morse Lemma

In this section, we prove an isotopy lemma (Theorem 3.1) adapted to the local study of stratified Morse functions. As consequences, we derive some local normal form statements, one of which (Corollary 3.2) may be called the stratified Morse lemma. The results of this section are very close to those of H. King in [7] and [8]. We continue with a Whitney stratified smooth manifold $(X, \mathcal{S})$. As in Section 1.2, we denote by Λ^0 the set of generic conormal vectors to $\mathcal{S}$. We also let $\Lambda^0_S = \Lambda^0 \cap \Lambda_S$, for each $S \in \mathcal{S}$.

Theorem 3.1 *Let $S \in \mathcal{S}$ with $\dim S = s$. Let $B \subset S$ be a closed s-ball smoothly embedded in S, with interior $B^\circ \subset B$ and a fixed point $a \in B^\circ$. Let $\mathcal{U} \subset X$ be an open neighborhood of B, let $p : \mathcal{U} \to S$ be a smooth submersion restricting to the identity on $\mathcal{U} \cap S$, and let $f : \mathcal{U} \to \mathbb{R}$ be a smooth function such that $f|_{S\cap\mathcal{U}} = 0$ and $d_b f \in \Lambda^0_S$ for every $b \in B$. Write $N = p^{-1}(a)$. Then there exist an open set $U \subset \mathcal{U}$ with $U \cap S = B^\circ$, control data on U which is p-compatible on U and f-compatible on $U \setminus S$, and a controlled homeomorphism $\phi : (U \cap N) \times B^\circ \to U$ such that:*

(i) $\phi\,(x, a) = x$ for every $x \in U \cap N$;

(ii) $p \circ \phi\,(x, y) = y$ for every $x \in U \cap N$ and $y \in B^\circ$.

(iii) $f \circ \phi\,(x, y) = f(x)$ for every $x \in U \cap N$ and $y \in B^\circ$.

Corollary 3.2 *Let $f : X \to \mathbb{R}$ be a smooth function with a stratified Morse critical point a, lying in a stratum S, let $\mathcal{U} \subset X$ be a neighborhood of a, and let $p : \mathcal{U} \to S$ be a smooth submersion restricting to the identity on $\mathcal{U} \cap S$. Write $N = p^{-1}(a)$. Then there exist a smaller neighborhood $U \subset \mathcal{U}$ of a, control data on U which is p-compatible on U and f-compatible on $U \setminus S$, and a controlled homeomorphism $\phi : (U \cap N) \times (U \cap S) \to U$ such that:*

(i) $\phi(x, a) = x$ for every $x \in U \cap N$;

(ii) $p \circ \phi(x, y) = y$ for every $x \in U \cap N$ and $y \in U \cap S$;

(iii) $f \circ \phi(x, y) = f(x) + f(y) - f(a)$ for every $x \in U \cap N$ and $y \in U \cap S$.

Proof: Apply Theorem 3.1 to the function $f - f \circ p$. □

Remark 3.3 Corollary 3.2 can be used to give a short proof of Goresky and MacPherson's product theorem for Morse data, as stated in Chapter 1 of [4] (Theorem STM, part B). This was earlier observed by H. King in [7] and [8]. We should also mention that a different short proof of Goresky and MacPherson's result has appeared recently in [6].

Corollary 3.4 *Let S be a stratum. Suppose we have two points $a, \hat{a} \in S$, two normal slices $N, \hat{N}$ to S passing through these points, and two functions $f : N \to \mathbb{R}$ and $\hat{f} : \hat{N} \to \mathbb{R}$ with $f(a) = \hat{f}(\hat{a}) = 0$. Assume that the differentials $d_a f$ and $d_{\hat{a}} \hat{f}$ are both in Λ_S^0 and, moreover, in the same path-component of Λ_S^0. Then there exist open neighborhoods $U \subset N$ and $\hat{U} \subset \hat{N}$ of a and $\hat{a}$, control data on U and $\hat{U}$, f- (resp. $\hat{f}$-) compatible on $U \setminus \{a\}$ (resp. $\hat{U} \setminus \{\hat{a}\}$), and a controlled homeomorphism $\phi : U \to \hat{U}$ such that $f = \hat{f} \circ \phi$.*

Proof: Apply Theorem 3.1 to a suitable function on the product $X \times \mathbb{R}$. □

Remark 3.5 As another corollary of Theorem 3.1, we note the fact that stratified Morse functions are topologically locally stable in the same sense as the ordinary Morse functions. In other words, a stratified Morse function is (locally) right-equivalent, by a controlled homeomorphism, to any nearby function with the same critical value. We omit the precise statement.

The proof of Theorem 3.1 is based on the following lemma.

Lemma 3.6 *In the situation of Theorem 3.1, fix a Riemannian metric g on X. Let $r : \mathcal{U} \to \mathbb{R}_{\geq 0}$ be the distance-to-S function. Let $\alpha = (f, r) : \mathcal{U} \setminus S \to \mathbb{R} \times \mathbb{R}_+$. Then there exist an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of B and number $k > 0$, such that $(p, \alpha) : \mathcal{U} \setminus S \to S \times \mathbb{R} \times \mathbb{R}_+$ is a stratified submersion on the set $\{x \in \mathcal{U}' \setminus S \mid |f(x)| \leq k \cdot r(x)\}$.*

Proof: Suppose the lemma is false. Let $\Sigma \subset \mathcal{U} \setminus S$ be the stratified critical locus of the map (p, α). Then there exists a sequence $\{x_i \in \Sigma\}$, converging to a point $b \in B$, such that

$$\lim_{i\to\infty} \frac{f(x_i)}{r(x_i)} = 0. \qquad (1)$$

By passing if necessary to a subsequence, we can assume that all the $\{x_i\}$ lie in the same stratum R, and that there exists a limit $\Delta = \lim T_{x_i}R \subset T_bX$. But then, combining equation (1) with the Whitney conditions for the pair (S, R), we may conclude that the differential d_bf annihilates Δ. This, however, contradicts the genericity assumption $d_bf \in \Lambda^0_S$. □

Proof of Theorem 3.1: Without loss of generality, we may assume that X is the total space of a vector bundle $X \to M$, that S is the zero section, and that $f : \mathcal{U} \to \mathbb{R}$ is the restriction of a fiber-wise linear function $\tilde{f} : X \to \mathbb{R}$. Fix a Euclidean structure in the bundle $X \to M$, and let $r : X \to \mathbb{R}_{\geq 0}$ be the associated norm. Let $\alpha = (f, r) : \mathcal{U} \setminus S \to \mathbb{R} \times \mathbb{R}_+$, and apply Lemma 3.6 to produce a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of B and a number $k > 0$.

Step 1. Let $R_k = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}_+ \mid |\xi| \leq k \cdot \eta\}$. There exists a smooth, R_+-equivariant function $\tilde{\rho} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tilde{\rho}(\xi, \eta) = |\xi|$ for all $(\xi, \eta) \notin R_k$.

Step 2. Let $\rho : \mathcal{U}' \setminus S \to \mathbb{R}_+$ be the composition $\rho = \tilde{\rho} \circ \alpha$. Then ρ is a quasi-distance function for S.

Step 3. There exist control data $\{U_T, \Pi_T, \rho_T\}$ on $\mathcal{U}'$ such that $U_S = \mathcal{U}'$, $\Pi_S = p|_{\mathcal{U}'}$, $\rho_S = \rho$, and $\{U_T, \Pi_T, \rho_T\}$ is α-compatible on some open set containing $\mathcal{U}' \cap \alpha^{-1}(R_k)$.

Step 4. Every smooth vector field V on B extends to a controlled vector field $\tilde{V}$ on $\mathcal{U}' \cap p^{-1}(B)$, compatible with $\{U_T, \Pi_T, \rho_T\}$ and satisfying $d\alpha(\tilde{V}_x) = 0$ for every $x \in \mathcal{U}' \cap p^{-1}(B) \cap \alpha^{-1}(R_k)$.

Step 5. The homeomorphism ϕ is constructed by integrating the $\tilde{V}_i$ for a suitable collection $\{V_i\}_{i=1}^s$ of vector fields on B. □

4 Construction of the Set K

We now begin the proof of Theorem 1.5. It is based on the following lemma.

Lemma 4.1 *In the situation of Theorem 1.5, there exist a covector $l \in \Delta$ (which we regard as a map $l : X \to \mathbb{C}$), a complete linear flag $\{p\} = F_0 \subset F_1 \subset \ldots \subset F_d = X$, a closed ball $B \subset X$ around p, and an algebraic Whitney stratification $\mathcal{X}$ refining $\mathcal{S}$, such that the following three conditions hold.*

(i) For $S \in \mathcal{X}$, if $p \notin \bar{S}$ then $\bar{S} \cap B = \emptyset$.

(ii) Let $Q_i = X/F_{d-i}$ and $\pi_i : V \to Q_i$ be the projection. Then for every $S \in \mathcal{X}$ of dimension $i > 0$, the map $\pi_{i-1} \oplus l : S \cap B \to Q_{i-1} \oplus \mathbb{C}$ has full rank.

(iii) Let X_i be the union of all $S \in \mathcal{X}$ with $\dim S \leq i$. Then for every $i = 1, \ldots, d$, the intersection $X_i \cap F_{d-i+1} \cap \operatorname{Ker} l \cap B = \{p\}$.

Assuming Lemma 4.1 for the moment, we take f in Theorem 1.5 to be the real part of the covector l, and we let the ball B to be the same as in the lemma. We now describe the subset $K \subset B$. It is defined as a union $K = K^+ \cup K^-$, where K^+ and K^- are constructed inductively. Let $K_0^+ = K_0^- = \{p\}$. Suppose now $1 \le i \le d$, and the sets $K_{i-1}^{\pm}$ have been constructed. We set

$$K_i^+ = \{x \in X_i \cap B \mid \operatorname{Im} l(x) = 0 \ \&\ \exists\, y \in K_{i-1}^+ : \pi_{i-1}(x) = \pi_{i-1}(y),\ f(x) \ge f(y)\},$$

$$K_i^- = \{x \in X_i \cap B \mid \operatorname{Im} l(x) = 0 \ \&\ \exists\, y \in K_{i-1}^- : \pi_{i-1}(x) = \pi_{i-1}(y),\ f(x) \le f(y)\},$$

where $\operatorname{Im} l : X \to \mathbb{R}$ is the imaginary part of l. Lastly, we set $K^{\pm} = K_d^{\pm}$. It is clear that K is a closed, real semi-algebraic subset of B.

Lemma 4.2 *With these definitions, conditions (i) and (ii) of Theorem 1.5 are satisfied.*

Proof: To check condition (i), note that $K^{\pm} \cap X_i = K_i^{\pm}$. We now prove by induction on i that $\dim_{\mathbb{R}} K_i^{\pm} \le i$. Indeed, case $i = 0$ is trivial, and the induction step follows from the definition of $K_i^{\pm}$ and condition (ii) of Lemma 4.1.

To check condition (ii), we argue by contradiction. Suppose the set $K^+ \cap f^{-1}(0)$ is larger than $\{p\}$ (the case of K^- is of course analogous). Let i be the smallest integer such that $K_i^+ \cap f^{-1}(0)$ contains a point $x \ne p$. By construction, there is a $y \in K_{i-1}^+$ with $\pi_{i-1}(x) = \pi_{i-1}(y)$ and $f(x) \ge f(y)$. By the minimality of i, we must have $y = p$ and, therefore, $x \in F_{n-i+1}$. But x is also in $\operatorname{Ker} l$, since $\operatorname{Re} l(x) = 0$ by assumption and $\operatorname{Im} l(x) = 0$ by the construction of K_i^+. Thus we have a contradiction with condition (iii) of Lemma 4.1. □

Proof of Lemma 4.1: Without loss of generality, we can assume that every stratum of $\mathcal{S}$ is connected (i.e., irreducible). All stratifications in this proof will have connected strata, so we can refer to "the generic point of a stratum" with no ambiguity. We proceed inductively, starting with $i = 1$ and going up to $i = d$, to construct the following four things:

- the i-plane $F_i \subset V$;
- the set $X_{d-i} \subset X$;
- a covector $l_i \in \Delta$;
- an (algebraic, Whitney) refinement $\mathcal{X}_i$ of the stratification $\mathcal{S}$.

When the process is complete, we will take $l = l_d$ and $\mathcal{X} = \mathcal{X}_d$. After the i-th step of the construction, the following conditions will be satisfied:

(1) X_{d-i} is the union of $S \in \mathcal{X}_i$ with $\dim S \le d - i$;

(2) $\dim X_{d-i} \cap \operatorname{Ker} l_i \le d - i - 1$;

(3) if $i > 1$, every $S \in \mathcal{X}_{i-1}$ with $\dim S \ge d - i + 2$ is also a stratum of $\mathcal{X}_i$;

(4) if $i > 1$, we have $l_i|_{F_i} = l_{i-1}|_{F_i}$;

(5) p is an isolated point of $F_i \cap X_{d-i}$;

(6) for every $S \in \mathcal{X}_i$ with $\dim S = d - i$ and $p \in \bar{S}$, the projection $\pi_{d-i} : S \to Q_{d-i}$ has full rank near p;

(7) p is an isolated point of $F_i \cap X_{d-i+1} \cap \operatorname{Ker} l_i$;

(8) for every $S \in \mathcal{X}_i$ with $\dim S = d-i+1$ and $p \in \bar{S}$, the map $\pi_{d-i} \oplus l_i : S \to Q_i \oplus \mathbb{C}$ has full rank near p.

Note that (3) and (4) ensure that (7) and (8) will continue to hold if we replace l_i by l_d and $\mathcal{X}_i$ by $\mathcal{X}_d$. Thus, our construction will prove the lemma.

As a base step of the induction, we take $\mathcal{X}_1 = \mathcal{S}$, so X_{d-1} is the union of all but the largest stratum of $\mathcal{S}$. Let $C(X_{d-1})$ be the normal cone of X_{d-1} at p. We take F_1 to be any line not contained in $C(X_{d-1})$, and l_1 to be any covector in Δ which does not vanish on F_1. Conditions (1)–(8) for $i = 1$ are clearly satisfied.

Assume now $i > 1$, and the first $i-1$ steps of the construction have been completed. To select the plane F_i, we consider two cones in $Q_{d-i+1} = X/F_{i-1}$. First, let X'_{d-i} be the union of all $S \in \mathcal{X}_{i-1}$ with $\dim S \leq d-i$, and let C_1 be the normal cone of the image $\pi_{d-i+1}(X'_{d-i})$ at p. It is a proper, closed cone in Q_{d-i+1}. Second, let C_2 be the normal cone at p of $\pi_{d-i+1}(\operatorname{Ker} l_{i-1} \cap X_{d-i+1})$. By part (2) of the induction hypothesis, this too is a proper, closed cone in Q_{d-i+1}. We now choose any line $L \subset Q_{d-i+1}$ not contained in $C_1 \cup C_2$, and set $F_i = \pi_{d-i+1}(L) \subset V$. This also defines the projection $\pi_{d-i} : V \to Q_{d-i}$. Let Σ be the set of all $S \in \mathcal{X}_{i-1}$ with $\dim S = d-i+1$ and $p \in \bar{S}$.

Claim 1: For every $S \in \Sigma$, the map $\pi_{d-i} \oplus l_{i-1} : S \to Q_{d-i} \oplus \mathbb{C}$ has full rank at the generic point of S.

Proof: Suppose the claim is false. Consider the intersection $\Gamma = F_i \cap S$. By part (6) of the induction hypothesis, there is an open neighborhood $U \subset X$ of p, such that $\Gamma \cap U$ is a smooth curve, cut out transversely as the zero set of $\pi_{d-i}|_{S \cap U}$. By part (5) of the induction hypothesis, and because the line L in the construction of F_i was chosen not to lie in the cone C_1, we have $p \in \bar{\Gamma}$. Further, since L was also chosen not to lie in the cone C_2, we can assume that there are no critical points of $l_{i-1}|_\Gamma$ in U. This means that the map $\pi_{d-i} \oplus l_{i-1} : S \to Q_{d-i} \oplus \mathbb{C}$ has full rank at every point of $\Gamma \cap U$. $\square$

For $S \in \Sigma$, let S° be the part of S where the map $\pi_{d-i} \oplus l_{i-1} : S \to Q_{d-i} \oplus \mathbb{C}$ has full rank. We set

$$X_{d-i} = X'_{d-i} \cup \bigcup_{S \in \Sigma} S \setminus S^\circ.$$

It is clear from the proof of Claim 1 that condition (5) is satisfied.

Claim 2: Let $S \in \Sigma$, and let $T \subset S \setminus S^\circ$ be an irreducible, smooth, locally closed subvariety with $\dim T = d-i$ and $p \in \bar{T}$. Then the restriction $\pi_{d-i}|_T$ has full rank at the generic point of T.

Proof: Suppose the claim is false. Then we have $\dim \pi_{d-i}(\bar{T}) < d-i$. Therefore, every fiber of $\pi_{d-i}|_{\bar{T}}$ must have positive dimension at every point. But it is clear from the proof of Claim 1 that p is an isolated point of $F_i \cap \bar{T}$. $\square$

We are now ready to describe the stratification $\mathcal{X}_i$. We take every stratum $S \in \mathcal{X}_{i-1}$ with $\dim S > d - i$ and $S \notin \Sigma$ to be also a stratum of $\mathcal{X}_i$. This ensures that conditions (1) and (3) are satisfied. For $S \in \Sigma$, we take each irreducible component of S° to be a stratum of $\mathcal{X}_i$. It remains to stratify the set X_{d-i}. By Claim 2, this can be done in such a way that the resulting stratification $\mathcal{X}_i$ is a Whitney refinement of $\mathcal{X}_{i-1}$, and condition (6) is satisfied.

The last thing to construct is the covector l_i. To satisfy condition (2) we must ensure that l_i does not vanish identically on any of the $(d-i)$-dimensional strata of $\mathcal{X}_i$. We take $l_i = l_{i-1} + h \circ \pi_{d-i}$, where $h : Q_{d-i} \to \mathbb{C}$ is a small linear functional in general position. It is easy to check using condition (5) that, for a suitable choice of h, we will have $l_i \in \Delta$ and condition (2) will hold.

This completes the construction of the quadruple $\{F_i, X_{d-i}, l_i, \mathcal{X}_i\}$. We have already remarked that conditions (1), (2), (3), (5), and (6) are satisfied. Condition (4) is clear from the definition of l_i. Condition (7) follows from the fact that the line L in the construction of F_i was chosen not to lie in the cone C_2, combined with condition (4) and part (5) of the induction hypothesis. Finally, condition (8) follows from the definition of the loci S° $(S \in \Sigma)$, again combined with condition (4). $\square$

5 Construction of the Flows

In this section, we complete the proof of Theorem 1.5 by constructing the ∇f-like vector fields appearing in part (iii) of that theorem. We begin with some preparations (keeping the notation of Section 4).

Theorem 1.5 stipulates that the vector field V should be $\mathcal{S}$-preserving, i.e., weakly controlled with respect to some control data on $(X, \mathcal{S})$. However, by Proposition 2.14, it suffices to construct, for each $\mathcal{U} \supset K$, a vector field V which is $\mathcal{X}$-preserving, instead. We therefore fix some control data on $(X, \mathcal{X})$, subject to the only condition that the quasi-distance function $\rho_{\{p\}}$ is the standard Euclidean distance to p. All weakly controlled vector fields in the rest of this section will be compatible with this control data.

Let $\mathcal{X}'$ be the set of all $S \in \mathcal{X}$ with $S \cap B \neq \emptyset$ and $\dim S > 0$. For every $S \in \mathcal{X}'$, with $\dim S = i$, let V^S be the unique smooth vector field on $S \cap B$ satisfying the following properties:

(a) the (standard, Euclidean) norm $\|V^S_x\| = \|x\|$ for every $x \in S \cap B$;
(b) V^S preserves the projection $\pi_{i-1} : S \cap B \to Q_{i-1}$;
(c) V^S preserves the imaginary part $\operatorname{Im} l : S \cap B \to \mathbb{R}$;
(d) $V^S_x f > 0$ for every $x \in S \cap B$.

For every $S \in \mathcal{X}'$, we fix a continuous, controlled extension $\tilde{V}^S$ of V^S to some neighborhood U_S of $S \cap B$. The vector field V will be constructed by "patching together" the $\tilde{V}^S$.

Without loss of generality, we may assume that $U_S \cap T = \emptyset$, for all distinct $S, T \in \mathcal{X}'$ with $\dim T \leq \dim S$. Let $\tilde{U}_S$ be the union of all $T \in \mathcal{X}'$ with $\dim T > \dim S$ or

$T = S$. We say that $\phi : \tilde{U}_S \to [0,1]$ is a cut-off function for $S \cap B$ if $\mathrm{supp}(\phi) \subset U_S$ and $\phi^{-1}(1)$ contains a neighborhood of $S \cap B$. Given two cut-off functions ϕ, ψ for $S \cap B$, we write $\phi \prec \psi$ if $\mathrm{supp}(\psi) \subset \phi^{-1}(1)$.

Now, let Φ be the set of all collections $\bar{\phi} = \{\phi_S\}_{S \in \mathcal{X}'}$, where ϕ_S is a cut-off function for $S \cap B$. Given $\bar{\phi}, \bar{\psi} \in \Phi$, we write $\bar{\phi} \prec \bar{\psi}$ if $\phi_S \prec \psi_S$ for all $S \in \mathcal{X}'$. It is easy to see that $(\Phi, \prec)$ is a directed set:

$$\forall\, \bar{\phi}_1, \bar{\phi}_2 \in \Phi \;\; \exists\, \bar{\phi}_3 \in \Phi : \; \bar{\phi}_1 \prec \bar{\phi}_3, \; \bar{\phi}_2 \prec \bar{\phi}_3.$$

Fix an ordering $\mathcal{X}' = \{S_1, \ldots, S_n\}$ such that $\dim S_i \leq \dim S_j$ for $i \leq j$. For every $\bar{\phi} \in \Phi$, we define a vector field $V = V(\bar{\phi})$ on B as follows. Let $V_p = 0$, and use the formula

$$V = \phi_{S_1} \cdot \tilde{V}^{S_1} + (1 - \phi_{S_1}) \cdot (\phi_{S_2} \cdot \tilde{V}^{S_2} + (1 - \phi_{S_2}) \cdot (\phi_{S_3} \cdot \tilde{V}^{S_3} + \ldots + (1 - \phi_{S_{n-1}}) \cdot V^{S_n}) \ldots)$$

on $B \setminus \{p\}$. This formula should be parsed left to right, and evaluation should stop as soon as one of the expressions $(1 - \phi_{S_i})$ is found to be zero. In this way, we will never have to evaluate one of the ϕ_{S_j} or $\tilde{V}^{S_k}$ at a point where it is not defined. Also, note the missing ˜ over V^{S_n}; it is not needed because S_n is the open stratum of $\mathcal{X}$. It is not hard to check that there is a $\bar{\phi}_0 \in \Phi$ such that, for every $\bar{\phi} \in \Phi$ with $\bar{\phi}_0 \prec \bar{\phi}$, the vector field $V(\phi)$ is ∇f-like. Theorem 1.5 is a consequence of the following claim.

Claim 1: For every open $\mathcal{U} \supset K$, there exists a $\bar{\phi}_0 \in \Phi$ such that, for every $\bar{\phi} \in \Phi$ with $\bar{\phi}_0 \prec \bar{\phi}$, we have $M^{\pm}_{V(\phi)}(p) \subset \mathcal{U}$.

Claim 1, in turn, follows from the following.

Claim 2: For every $x \in B \setminus K$, there is a neighborhood U_x of x and a $\bar{\phi}_0 \in \Phi$ such that, for every $\bar{\phi} \in \Phi$ with $\bar{\phi}_0 \prec \bar{\phi}$, we have $M^{\pm}_{V(\phi)}(p) \cap U_x = \emptyset$.

Claim 2 is readily proved by induction on the dimension of the stratum containing x, using the definition of K and the following.

Claim 3: Fix an $i \in \{0, \ldots, d-1\}$, and let $A \subset B$ be a compact set such that $A \cap X_i = \emptyset$. Then for every $\epsilon > 0$, there is a $\bar{\phi}_0 \in \Phi$ such that, for every $\bar{\phi} \in \Phi$ with $\bar{\phi}_0 \prec \bar{\phi}$ and every $a \in A$, we have the following estimates:

$$\|d\pi_i(V(\phi)_a)\| < \epsilon \cdot df(V(\phi)_a),$$

$$|\,\mathrm{Im}\; dl(V(\phi)_a)| < \epsilon \cdot df(V(\phi)_a).$$

Lastly, Claim 3 follows from the fact that the $\tilde{V}^S$ in the construction of $V(\phi)$ were chosen to be continuous extensions of the vector fields V^S satisfying conditions (b)-(d). This completes the proof of Theorem 1.5.

6 Self-Indexing

In this section, we give a proof of Theorem 1.6. It follows the same scheme as the proof of Theorem 1.1 outlined in Section 1.1.

Definition 6.1 *Let $(X, \mathcal{S})$ be a compact, Whitney stratified C^∞ manifold. A stratified subset $A \subset X$ is a closed subset presented as a finite disjoint union $A = \bigcup A_i$ so that each A_i is a smooth submanifold of one of the strata of $\mathcal{S}$, and the frontier $\overline{A_i} \setminus A_i$ is a union of several of the A_j with $\dim A_j < \dim A_i$.*

Definition 6.2 *Let $(X, \mathcal{S})$ be a compact, Whitney stratified C^∞ manifold, with fixed control data. A time-dependent controlled vector field $\{V_t\}_{t\in[0,1]}$ on $(X, \mathcal{S})$ is a controlled vector field defined in some neighborhood of $X \times [0, 1] \subset X \times \mathbb{R}$, whose component in the $\mathbb{R}$-direction is identically zero.*

Lemma 6.3 *Let $(X, \mathcal{S})$ be as in Definition 6.2, and let $A, B \subset X$ be two stratified subsets. Assume that for every $S \in \mathcal{S}$, we have:*

$$\dim(A \cap S) + \dim(B \cap S) < \dim S.$$

Then there exists a time-dependent controlled vector field $\{V_t\}_{t\in[0,1]}$ on X whose time-1 flow $\psi_{V,1} : X \to X$ satisfies $\psi_{V,1}(A) \cap B = \emptyset$.

Proof: The restriction of V_t to the i-skeleton of $\mathcal{S}$ is constructed by induction on i. The induction step is an application of the general position in manifolds. See [12, §3] for a proof of a much more general stratified general position result. $\square$

Theorem 1.6 follows from Lemmas 6.4 and 6.5 below.

Lemma 6.4 *Let $(X, \mathcal{S})$ be a Whitney stratified complex algebraic variety, and let $g : X \to \mathbb{R}$ be a proper Morse function. Let $I \subset \mathbb{R}$ be an open interval whose preimage contains exactly two critical points: $g^{-1}(I) \cap \Sigma_f = \{p, q\}$. Assume that $\mathrm{index}_g(p) \le \mathrm{index}_g(q)$ and $g(p) \ge g(q)$. Then for every $a, b \in I$, there is a Morse function $f : X \to \mathbb{R}$ such that:*
(i) $f^{-1}(I) = g^{-1}(I)$, and $f = g$ outside of some compact subset of $g^{-1}(I)$;
(ii) $f^{-1}(I) \cap \Sigma_f = \{p, q\}$;
(iii) in some neighborhood of p we have $f(x) = g(x) + a - g(p)$;
(iv) in some neighborhood of q we have $f(x) = g(x) + b - g(q)$.

Lemma 6.5 *Let $(X, \mathcal{S})$ and $g : X \to \mathbb{R}$ be as in Lemma 6.4, and let $I \subset \mathbb{R}$ be an open interval such that all the critical points in $g^{-1}(I)$ have the same index. Then for every $c \in I$, there is a Morse function $f : X \to \mathbb{R}$ such that:*
(i) $f^{-1}(I) = g^{-1}(I)$, and $f = g$ outside of some compact subset of $g^{-1}(I)$;
(ii) $f^{-1}(I) \cap \Sigma_f = g^{-1}(I) \cap \Sigma_g$;
(iii) near each $p \in f^{-1}(I) \cap \Sigma_f$ we have $f(x) = g(x) + c - g(p)$.

The proofs of Lemmas 6.4 and 6.5 are similar; we will only give the first.

Proof of Lemma 6.4: *Step 1:* The case when $g(p) = g(q)$ is obvious, so we assume that $g(p) > g(q)$. Pick two numbers $c < d$ from the interval $(g(q), g(p))$. Let $D = g^{-1}[c, d]$. Fix control data $\{U_S, \Pi_S, \rho_S\}$ on $(X, \mathcal{S})$ which is g-compatible in some neighborhood of D.

Claim 1: There exist a ∇g-like vector field V^D on D, compatible with $\{U_S, \Pi_S, \rho_S\}$, and a pair of stratified subsets $A, B \subset g^{-1}(d)$ such that:

(i) for every $S \in \mathcal{S}$ with $\dim_{\mathbb{C}} S = s$, we have:

$$\dim_{\mathbb{R}}(A \cap S) < s - \mathrm{index}_g(q);$$

$$\dim_{\mathbb{R}}(B \cap S) < s + \mathrm{index}_g(p).$$

(ii) for every pair of open neighborhoods $\mathcal{U}_A, \mathcal{U}_B \subset g^{-1}(d)$ of A and B, there is a ∇g-like vector field V on X with $V|_D = V^D$ such that:

$$(M_V^+(q) \cap g^{-1}(d)) \subset \mathcal{U}_A;$$

$$(M_V^-(p) \cap g^{-1}(d)) \subset \mathcal{U}_B.$$

Claim 1 follows by putting together Theorem 1.5, Proposition 2.14, and Corollaries 3.2 and 3.4.

Step 2: Using Lemma 6.3 to modify the vector field V^D, we can strengthen Claim 1 as follows.

Claim 2: The sets A and B in Claim 1 can be chosen to satisfy $A \cap B = \emptyset$.

This immediately implies that there exists a ∇g-like vector field V on X, compatible with $\{U_S, \Pi_S, \rho_S\}$, such that $M_V^-(p) \cap M_V^+(q) = \emptyset$.

Step 3: Now, choose a smooth function $\phi : g^{-1}(d) \to [0, 1]$ such that:

(a) the control data $\{U_S, \Pi_S, \rho_S\}$ on $(X, \mathcal{S})$ restricts to ϕ-compatible control data on $g^{-1}(d)$ (with the stratification induced from $\mathcal{S}$);

(b) there is a neighborhood $\mathcal{U}_A \subset g^{-1}(d)$ of $M_V^+(q) \cap g^{-1}(d)$ such that $\phi|_{\mathcal{U}_A} = 0$;

(c) there is a neighborhood $\mathcal{U}_B \subset g^{-1}(d)$ of $M_V^-(p) \cap g^{-1}(d)$ such that $\phi|_{\mathcal{U}_b} = 1$.

It is not hard to check that ϕ extends uniquely to a smooth function $\tilde{\phi} : g^{-1}(I) \to [0, 1]$, which is constant along the flow lines of V.

The function f is now easy to construct by setting $f|_{g^{-1}(I)} = \tilde{f} \circ \alpha$, where $\alpha = (g, \phi) : g^{-1}(I) \to I \times [0, 1]$ and $\tilde{f} : I \times [0, 1] \to I$ is a suitable smooth function of two variables. $\square$

This completes the proof of Theorem 1.6.

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